MATH 2020 A
Fall 2020

# Lecture 23

- divergence thm (cont'd)
- differential forms

Last time we discussed the divergence thm:

$$\iiint_{\Omega} \operatorname{div} \vec{F}\, dV = \oiint_{S} \vec{F}\cdot\hat{n}\, d\sigma$$

where $\Omega \subseteq \mathbb{R}^3$ and $\partial\Omega = S$, $\hat{n}$ outer unit normal.

$\iint_S \vec{F}\cdot\hat{n}$ is the (outward) flux of $\vec{F}$ across $S$. Like in Stokes' thm we used it to obtain the measure of rotation at a point $(x,y,z)$ around $\xi$-axis by

$$\nabla\times\vec{F}(x,y,z)\cdot\hat{\xi} = \lim_{\varepsilon\to 0}\frac{1}{|C_\varepsilon|}\oint_{C_\varepsilon}\vec{F}\cdot\hat{t}\, ds$$

($C_\varepsilon$ circle center at $(x,y,z)$, radius $\varepsilon$, lying on the plane so that $\hat{\xi}$ pts to its "$+z$"-axis.), we take a ball $B_\varepsilon$ center at $(x,y,z)$ and consider

$$\frac{1}{|B_\varepsilon|}\iiint_{B_\varepsilon}\operatorname{div}\vec{F}\, dV = \frac{1}{|B_\varepsilon|}\oiint_{S_\varepsilon}\vec{F}\cdot\hat{n}\, d\sigma, \quad S_\varepsilon = \partial B_\varepsilon.$$

$$\therefore \operatorname{div}\vec{F}(x,y,z) = \lim_{\varepsilon\to 0}\frac{1}{|B_\varepsilon|}\oiint_{S_\varepsilon}\vec{F}\cdot\hat{n}\, d\sigma$$

which gives a meaning to div $\vec{F}$.

Divergence then has $\infty$-many applications! especially in the study of partial differential equations. Let's look at one example.

Image the space is occupied with particles that are moving according to some physical law (we don't need to specify it) as long as it is continuous. Its density is denoted by

$$\delta(x, y, z, t)$$

which varies in time. Assume that neither mass is generated nor vanishes, we derive an equation its velocity $\vec{u}(x, y, z, t)$ should satisfy.

Fix an arbitrary region $\Omega \subseteq \mathbb{R}^3$. the total mass at $t$ is

$$M(t) = \iiint_{\Omega} \delta(x, y, z, t)\, dV(x, y, z)$$

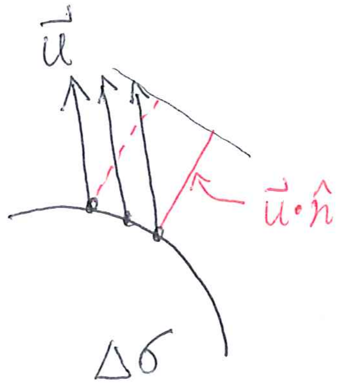
So, the rate of change in mass:

$$\frac{dM(t)}{dt} = \lim_{\Delta t \to 0} \frac{M(t+\Delta t) - M(t)}{\Delta t}$$

$$= \iiint_{\Omega} \frac{\partial \delta}{\partial t}(x, y, z)\, dV(x, y, z). \qquad (1)$$

On the other hand, since there is no gain nor loss of mass, the change of mass must take place at the boundary of $\Omega$.

Let $S = \partial M$ and $\Delta\sigma$ a small piece on $S$. In a small time $\Delta t$,




the volume of particles passing $\Delta\sigma$ in $\Delta t$ is

$$\vec{u}\cdot\hat{n}\,\Delta\sigma\,\Delta t$$

So the mass leaving this $\Delta\sigma$ in $\Delta t$ time is

$$\delta\vec{u}\cdot\hat{n}\,\Delta\sigma\,\Delta t$$

($\vec{u}\cdot\hat{n} > 0$ here) Using the idea of Riemann sum, the total mass that leaving $\Omega$ in $\Delta t$ is

$$\left(\iint_S \delta\vec{u}\cdot\hat{n}\,d\sigma\right)\Delta t, \quad \text{ie}$$

$$M(t+\Delta t) - M(t) \approx -\iint_S \delta\vec{u}\cdot\hat{n}\,d\sigma\,\Delta t, \quad \text{or}$$

$$\frac{M(t+\Delta t) - M(t)}{\Delta t} \approx -\iint_S \delta\vec{u}\cdot\hat{n}\,d\sigma$$

$$\therefore \quad \frac{dM}{dt} = -\iint_S \delta\vec{u}\cdot\hat{n}\,d\sigma$$

key point: $$= -\iiint_\Omega \operatorname{div}\delta\vec{u}\,dV \qquad (\text{div. thm})$$

$$① + ② : \quad \iiint_\Omega \left(\frac{\partial\delta}{\partial t} + \operatorname{div}\delta\vec{u}\right)dV = 0,$$

Since $\Omega$ is arbitiary, we get the equation of continuity:

$$\frac{\partial\delta}{\partial t} + \operatorname{div}(\delta\vec{u}) = 0,$$

which is a form of localized conservation of mass.

# Differential forms and Stokes' thm

We present a formal theory of differential forms in $\mathbb{R}^3$.

0-forms : just functions

1-forms : basic ones are $dx, dy, dz$.

All $k$-forms form a vector space.

Other higher order forms are obtained from the wedge product.

The wedge product produces a $(k+l)$-form from a $k$-form and an $l$-form.

Rules of wedge product :

(1) $f$ 0-form, $\omega$ $k$-form then $f\wedge\omega$ is a $k$-form (usually we drop $\wedge$ and write simply $f\omega$)

(2) for $\omega_1, \omega_2$ $k$-forms, $\eta$ $l$-form, $k, l \geq 1$,

(a) distributive : $(\omega_1+\omega_2)\wedge\eta = \omega_1\wedge\eta + \omega_2\wedge\eta$
$\eta\wedge(\omega_1+\omega_2) = \eta\wedge\omega_1 + \eta\wedge\omega_2$

(b) associative
$(\omega_1\wedge\omega_2)\wedge\omega_3 = \omega_1\wedge(\omega_2\wedge\omega_3)$

(c) antisymmetric : $\omega\wedge\eta = (-1)^{kl}\eta\wedge\omega$,
in particular,
$dx\wedge dy = -dy\wedge dx$
$dx\wedge dx = 0$, etc

Using (a)–(c), we see that every 1-form is of the form

$$\omega = f\,dx + g\,dy + h\,dz,$$

2-forms are

$$\omega = f\,dx\wedge dy + g\,dy\wedge dz + h\,dz\wedge dx$$

3-form $\omega = f\,dx \wedge dy \wedge dz$.
$k \geq 4$ forms all vanish.
Note that $0 \wedge \omega = 0$. (here $0$ could be $0dx + 0dy + 0dz$, or $0\,dx \wedge dy + 0\,dy \wedge dz + 0\,dz \wedge dx$, etc)

$\boxed{\text{e.g. 1}}$ Simplify $\omega \wedge \eta$, when $\omega = \varphi\,dx + \psi\,dz$,
$\eta = f\,dy + g\,dz$.

$$\omega \wedge \eta = (\varphi\,dx + \psi\,dz) \wedge (f\,dy + g\,dz)$$
$$= \varphi\,dx \wedge (f\,dy + g\,dz) + \psi\,dz \wedge (f\,dy + g\,dz)$$
$$= \varphi\,dx \wedge (f\,dy) + \varphi\,dx \wedge g\,dz + \psi\,dz \wedge f\,dy + \psi\,dz \wedge g\,dz$$
$$= \varphi f\,dx \wedge dy + \varphi g\,dx \wedge dz + \psi f\,dz \wedge dy + \psi g\,dz \wedge dz$$
$$= \varphi f\,dx \wedge dy - \varphi g\,dz \wedge dx - \psi f\,dy \wedge dz. \quad \#$$

$\boxed{\text{e.g. 2}}$ Simplify $\omega \wedge \eta$, $\omega = \varphi\,dx + \psi\,dz$
$\eta = h\,dx \wedge dy$.

$$\omega \wedge \eta = (\varphi\,dx + \psi\,dz) \wedge (h\,dx \wedge dy)$$
$$= \varphi\,dx \wedge (h\,dx \wedge dy) + \psi\,dz \wedge (h\,dx \wedge dy)$$
$$= \varphi(dx \wedge h\,dx) \wedge dy + \psi(dz \wedge h\,dx) \wedge dy$$
$$= \varphi h(dx \wedge dx) \wedge dy + \psi h(dz \wedge dx) \wedge dy$$
$$= \psi h\,dz \wedge dx \wedge dy = -\psi h\,dx \wedge dz \wedge dy$$
$$= \psi h\,dx \wedge dy \wedge dz. \quad \#$$

Exterior differentiation sends a k-form to a (k+1)-form. The rules are

(1) $df = f_x\,dx + f_y\,dy + f_z\,dz$

(2) $d(f\,dx + g\,dy + h\,dz)$
$\overset{\text{def}}{=} df \wedge dx + dg \wedge dy + dh \wedge dz$

(3) $d(f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx)$
$\overset{\text{def}}{=} df \wedge dx \wedge dy + dg \wedge dy \wedge dz + dh \wedge dz \wedge dx.$

(4) $d(f\,dx \wedge dy \wedge dz) = df \wedge dx \wedge dy \wedge dz = 0.$

Theorem 1 we have

(a) $d(f\,dx + g\,dy + h\,dz) = (-f_y + g_x)\,dx \wedge dy + (f_z - h_x)\,dz \wedge dx$
$+ (-g_z + h_y)\,dy \wedge dz,$

(b) $d(f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx)$
$= (f_z + g_x + h_y)\,dx \wedge dy \wedge dz.$

Pf: (a) $d(f\,dx + g\,dy + h\,dz) = df \wedge dx + dg \wedge dy + dh \wedge dz$
$= (f_x\,dx + f_y\,dy + f_z\,dz) \wedge dx + (g_x\,dx + g_y\,dy + g_z\,dz) \wedge dy$
$+ (h_x\,dx + h_y\,dy + h_z\,dz) \wedge dz$
$= f_y\,dy \wedge dx + f_z\,dz \wedge dx + g_x\,dx \wedge dy + g_z\,dz \wedge dy$
$+ h_x\,dx \wedge dz + h_y\,dy \wedge dz$
$= (-f_y + g_x)\,dx \wedge dy + (f_z - h_x)\,dz \wedge dx + (-g_z \wedge h_y)\,dy \wedge dz.$

(b) $d(f\,dx\wedge dy + g\,dy\wedge dz + h\,dz\wedge dx)$

$$= df\wedge dx\wedge dy + dg\wedge dy\wedge dz + dh\wedge dz\wedge dx$$
$$= (f_x dx + f_y dy + f_z dz)\wedge dx\wedge dy + (g_x dx + g_y dy + g_z dz)\wedge dy\wedge dz + (h_x dx + h_y dy + h_z dz)\wedge dz\wedge dx$$
$$= f_z\,dz\wedge dx\wedge dy + g_x\,dx\wedge dy\wedge dz + h_y\,dy\wedge dz\wedge dx$$
$$= (f_z + g_x + h_y)\,dx\wedge dy\wedge dz .$$

Theorem 2 For any form $\omega$, $d(d\omega) = 0$.

Pf: Exercise.

Under transformation: $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$,

$$dx = x_u\,du + x_v\,dv,$$
$$dy = y_u\,du + y_v\,dv,$$
$$dz = z_u\,du + z_v\,dv.$$

Similar situation holds for $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$.

We can "integrate" $k$-forms over a $k$-manifold (regions, surface, curves, etc).

Theorem 3 (Stokes' Thm) For $k$-form $\omega$ on a $k$-manifold $M$,

$$\int_M d\omega = \int_{\partial M} \omega ,$$ where $\partial M$ is the boundary of $M$, a $(k-1)$-manifold.

$k=3$

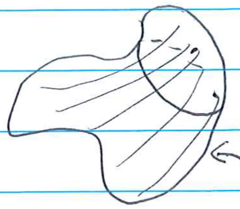
open / closed

$M=$ region, $\partial M=$ surface

$k=2$

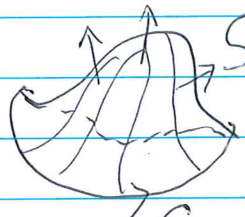
(oriented)

$M=$ surface, $\partial M=$ curves (or empty)




$M$ solid

$\partial M$ boundary




checking. First consider $\Omega \subseteq \mathbb{R}^3$ a region, $\partial\Omega$ a surface $S$.

For a 3-form $\eta$ on $\Omega$, $\eta = f\,dx\wedge dy\wedge dz$, so naturally we define

$$\int_\Omega \eta \overset{\text{def}}{=} \iiint_\Omega f\,dV \qquad (3)$$

To define $\int_{\partial\Omega}\omega$ we consider $D \xrightarrow{\vec{r}} S=\partial\Omega$ a regular parametr respect the normal on $S$

($\omega$ 2-form)

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v)$$

$$\omega = f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy \quad \text{(general form)}$$

$$\begin{aligned}
&= f(y_u\,du \wedge y_v\,dv)\wedge(z_u\,du+z_v\,dv) + \\
&\quad g(z_u\,du+z_v\,dv)\wedge(x_u\,du+x_v\,dv) + \\
&\quad h(x_u\,du+x_v\,dv)\wedge(y_u\,du+y_v\,dv) \\
&= f y_u z_v\,du\wedge dv + f y_v z_u\,dv\wedge du + g z_u x_v\,du\wedge dv \\
&\quad + g z_v x_u\,dv\wedge du + h x_u y_v\,du\wedge dv + h x_v y_u\,dv\wedge du \\
&= [f(y_u z_v - y_v z_u) + g(z_u x_v - x_u z_v) + h(x_u y_v - x_v y_u)]\,du\wedge dv
\end{aligned}$$

If we let $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$,

$$\omega = \underbrace{\vec{F} \cdot \vec{r}_u \times \vec{r}_v}_{\text{a fcn in } D} \, du\,dv$$

So, for a 2-form on $\partial\Omega$, we may define

$$\int_{\partial\Omega} \omega = \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\, d\sigma \quad , \ \hat{n} \text{ the chosen normal.} \qquad (4)$$

Checking Stokes' thm for 2-forms.

Let $\omega = f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy$ be a 2-form.
From Thm 1 (b)

$$d\omega = (h_z + f_x + g_y)\, dx\wedge dy\wedge dz \qquad \left(\text{Note the change of notations!}\right)$$

$d\omega$ is a 3-form, by (3)

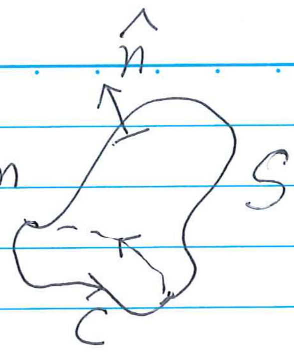
$$\int_{\Omega} d\omega = \iiint_{\Omega} \operatorname{div}\vec{F}\, dV \qquad (5)$$

By Divergence Thm, (5) and (4) are equal, ie

$$\iiint_{\Omega} \operatorname{div}\vec{F}\, dV = \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\, d\sigma$$

$$\therefore \quad \int_{\Omega} d\omega = \int_{\partial\Omega} \omega .$$

Checking Stokes Thm for 1-forms.




Let $\omega = f\,dx + g\,dy + h\,dz$ be 1-form

Let $[a,b] \xrightarrow{\vec{\gamma}(t)} C$ be a regular parametrization. Then

$C = \partial S$

$$dx = x'(t)\,dt$$
$$dy = y'(t)\,dt$$
$$dz = z'(t)\,dt$$

So $$f\,dx + g\,dy + h\,dz = (\underbrace{fx' + gy' + hz'}_{\text{a fcn of } t})\,dt$$

So we define

$$\int_{\partial S} \omega = \int_a^b \vec{F} \cdot \vec{\gamma}'\,dt$$

$$= \int_C \vec{F} \cdot d\vec{r} \qquad (6)$$

On the other hand, by Thm 1 (a)

$$d\omega = (-f_y + g_x)\,dx \wedge dy + (f_z - h_x)\,dz \wedge dx$$
$$+ (-g_z + h_y)\,dy \wedge dz.$$

According to (4), taking $f \leftrightarrow -g_z + h_y$, $g \leftrightarrow f_z - h_x$, $h \leftrightarrow -f_y + g_x$

$$\int_S d\omega = \iint_S \vec{H} \cdot \hat{n}\,d\sigma \qquad \text{where}$$

$$\vec{H} = (-g_z + h_y)\hat{i} + (f_z - h_x)\hat{j} + (-f_y + g_x)\hat{k}$$

$$= \nabla \times \vec{F},$$

$$\int_S d\omega = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma \qquad (5)$$

By stokes' thm, (6) = (5), that's

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \int_C \vec{F} \cdot d\vec{r}, \quad \text{so}$$

$$\int_S d\omega = \int_{\partial S} \omega.$$

For $\mathbb{R}^2$, there are 1-forms to consider. Let $D \subseteq \mathbb{R}^2$ be a simply connected open set, $\partial D = C$, a closed simple curve.

Let $\omega = f\,dx + g\,dy$ be a 1-form. Let $[a,b] \xrightarrow{\vec{\gamma}} C$ be a regular parametrization

$$f\,dx + g\,dy = (f x' + g y')\,dt$$

We define

$$\int_{\partial D} \omega = \int_a^b (f x' + g y')\,dt$$

$$= \int_C \vec{F} \cdot d\vec{r}, \qquad \vec{F} = f\hat{i} + g\hat{j}$$

$$d\omega = df \wedge dx + dg \wedge dy$$
$$= (f_x\,dx + f_y\,dy) \wedge dx + (g_x\,dx + g_y\,dy) \wedge dy$$
$$= f_y\,dy \wedge dx + g_x\,dx \wedge dy$$
$$= (g_x - f_y)\,dx \wedge dy .$$

So,

$$\int_D d\omega = \iint_D (g_x - f_y)\,dA .$$

By Green's thm, $\iint_D (g_x - f_y)\,dA = \int_C \vec{F} \cdot d\vec{r}$, so again

$$\int_D d\omega = \int_{\partial D} \omega . \quad \#$$

A complete treatment of differential forms and Stokes' theorem can be found in

M. Spivak, Calculus on Manifolds.